Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Geometry and Physics 51 (2004) 71–81

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Bijectivity of the canonical map for the non-commutative instanton bundle

F. Bonechi [a,b], N. Ciccoli [c], L. Dąbrowski [d], M. Tarlini [a,b,*]

[a] *INFN Sezione di Firenze, Firenze, Italy*
[b] *Dipartimento di Fisica, Università di Firenze, Firenze, Italy*
[c] *Dipartimento di Matematica, Università di Perugia, Italy*
[d] *Scuola Internazionale Superiore di Studi Avanzati, Trieste, Italy*

Received 9 June 2003; received in revised form 16 July 2003

Abstract

It is shown that the quantum instanton bundle introduced in [Commun. Math. Phys. 226 (2002) 419] has a bijective canonical map and is, therefore, a coalgebra Galois extension.

MSC: 16W30; 19D55; 58B32; 58B34; 81R50

JGP SC: Non-commutative geometry; Quantum groups

Keywords: Non-commutative geometry; Quantum groups; Quantum spheres; Instanton quantum bundle; Coalgebra Galois extension

1. Introduction

Since the beginning of the theory of quantum groups lot of efforts have been devoted to develop a full quantum analogue of the notion of principal bundle. For that purpose the key feature turns out to be a characterization of freeness and transitivity on the fiber of the group action through the bijectivity of the so-called canonical map. In [17] the relevant role of this map in the context of Hopf algebras was stressed and a theory of algebraic quantum principal bundles with a Hopf algebra as structure group was built up. A notion related to such a Hopf–Galois property but on the level of differential calculi and principal connections was partially used in [8,12]. Later on several examples of quantum principal

* Corresponding author.
E-mail addresses: bonechi@fi.infn.it (F. Bonechi), ciccoli@dipmat.unipg.it (N. Ciccoli), dabrow@sissa.it (L. Dąbrowski), tarlini@fi.infn.it (M. Tarlini).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2003.09.007

bundles have been shown to be indeed Hopf–Galois extensions and thus to fully deserve their name (see, e.g. [11] for a list).

However many interesting examples of deformations of principal bundles coming from quantum groups could not be expressed in that framework. For instance, a quantum homogeneous space is a Hopf–Galois extension only if it is a quotient by a quantum subgroup, which is a very rare case. It is therefore necessary to consider a more general object in place of the structure group. In [17] bundles with a *coalgebra structure group* obtained as quotient of Hopf algebras by a coideal and right ideal were first considered. Later on it was shown in [4] that all Podleś spheres can be obtained as quotients by a *coalgebra subgroup of* $\mathrm{SU}_q(2)$. Such results were further developed in [5,6,9] to the idea of coalgebra–Galois extensions and, more recently, in [7] into a possible definition of coalgebra principal bundles. It has to be stressed that the bijectivity of the canonical map retains a crucial role in the theory.

Approximately at the same time a *semiclassical* (i.e. Poisson) interpretation for this approach was given in [10] and [1], in terms of coisotropic subgroups of a Poisson–Lie group. The semiclassical point of view turned out to be very useful as it allowed in [2] to describe a quantum group version of the SU(2)-principal bundle $\mathbb{S}^7 \to \mathbb{S}^4$. In [3] a description of this bundle in terms of quantized enveloping algebras is given.

It is then quite natural to ask whether this construction can be considered an algebraic quantum principal bundle, the key point being a verification of the bijectivity of the canonical map, as explained. In this paper we will prove that property. Let us remark that the quantum instanton bundle is far more difficult to deal with than any other known example arising from quantum groups, for at least three different reasons: the base space is not a quantum homogeneous space (just a double coset of $\mathrm{U}_q(4)$), the structure group is only a coalgebra and, lastly, such coalgebra corresponds to a non-abelian group.

Once the Galois property is proven the next natural step to understand the geometry of this quantum principal bundle is to analyze the quantum vector bundles associated to corepresentations of the structure coalgebra. The bijectivity of the canonical map proven in this paper together with the results contained in [7] and in the work in preparation [16] allow to conclude that they are finitely generated and projective modules. In the last section we will explain this point.

Our results and the explicit expression of the K-homology generators given in [2] are all the needed ingredients to apply the character formula in [7] to compute the charges of these bundles, in analogy to the computations carried out for line bundles on Podleś sphere, both in the Hopf–Galois [13] and in the coalgebra–Galois case [14].

2. The quantum four-sphere Σ_q^4

In this section we recall the basic facts necessary to obtain the four-sphere Σ_q^4 ([2]). The algebra of polynomial functions $\mathcal{A}(\mathrm{U}_q(4))$ is generated by $\{t_{ij}\}_{ij=1}^4$, D_q^{-1} and the following relations:

$$t_{ik}t_{jk} = qt_{jk}t_{ik}, \qquad t_{ki}t_{kj} = qt_{kj}t_{ki}, \quad i < j, \qquad t_{i\ell}t_{jk} = t_{jk}t_{i\ell}, \quad i < j, \ k < \ell,$$
$$t_{ik}t_{j\ell} - t_{j\ell}t_{ik} = (q - q^{-1})t_{jk}t_{i\ell}, \quad i < j, \ k < \ell, \qquad D_qD_q^{-1} = D_q^{-1}D_q = 1, \tag{1}$$

where $D_q = \sum_{\sigma\in P_4}(-q)^{\ell(\sigma)} t_{\sigma(1)1}\cdots t_{\sigma(4)4}$ is the quantum determinant and P_4 is the group of 4-permutations. It is easy to see that D_q is central. The coproduct is

$$\Delta(t_{ij}) = \sum_k t_{ik}\otimes t_{kj}, \qquad \Delta(D_q) = D_q\otimes D_q. \tag{2}$$

For the usual definition of the antipode S see [15]; the compact real structure forces to choose $q\in\mathbb{R}$ and it reads

$$t_{ij}^* = S(t_{ji}), \qquad D_q^* = D_q^{-1}. \tag{3}$$

In the following we will denote $\kappa = *\circ S$.

The algebra of polynomial functions $\mathcal{A}(\mathbb{S}^7_q)$ on the quantum seven-sphere (see [18]) is generated as a $*$-algebra by $\{z_i = t_{4i}\}_{i=1}^4$, which verify the following relations:

$$\begin{aligned} &z_i z_j = q z_j z_i \quad (i<j), \qquad z_j^* z_i = q z_i z_j^* \quad (i\neq j),\\ &z_k^* z_k = z_k z_k^* + (1-q^2)\sum_{j<k} z_j z_j^*, \qquad \sum_{k=1}^4 z_k z_k^* = 1. \end{aligned} \tag{4}$$

The $\mathcal{A}(\mathrm{U}_q(4))$-coaction on $\mathbb{S}^7_q$ reads

$$\Delta(z_i) = \sum_j z_j\otimes t_{ji}, \qquad \Delta(z_i^*) = \sum_j z_j^*\otimes t_{ji}^*. \tag{5}$$

Define $\mathcal{R} = R\mathcal{A}(\mathrm{U}_q(4))$, where

$$\begin{aligned} R = \mathrm{Span}\{&t_{13}, t_{31}, t_{14}, t_{41}, t_{24}, t_{42}, t_{23}, t_{32}, t_{11}-t_{44}, t_{12}+t_{43}, t_{21}+t_{34}, t_{22}-t_{33},\\ &t_{11}t_{22}-qt_{12}t_{21}-1\}. \end{aligned}$$

It is easy to verify that $\mathcal{R}$ is a κ-invariant, right ideal, two sided coideal. In the sequel we shall denote $C := \mathcal{A}(\mathrm{U}_q(4))/\mathcal{R}$ the quotient and $r : \mathcal{A}(\mathrm{U}_q(4)) \to C$ the canonical projection.

By construction C is a coalgebra, a right $\mathcal{A}(\mathrm{U}_q(4))$-module and it inherits an involutive, antilinear map κ_C. In [2] it has been shown that C is isomorphic to $\mathcal{A}(\mathrm{SU}_q(2))$ as coalgebras, and that the isomorphism intertwines κ_C with $*\circ S$ on $\mathcal{A}(\mathrm{SU}_q(2))$. Using this isomorphism we could transfer to C the algebra structure of $\mathcal{A}(\mathrm{SU}_q(2))$ (and thus of a Hopf algebra) but the projection map $r : \mathcal{A}(\mathrm{U}_q(4)) \to C$ would not be a homomorphism, e.g. $r(t_{11}t_{43}) \neq r(t_{11})r(t_{43})$. In the rest of the paper we actually assume this point of view, and identify C with $\mathcal{A}(\mathrm{SU}_q(2))$. Although r does not respect the algebra structure C has a right $\mathcal{A}(\mathrm{U}_q(4))$-module structure, that will be important for us, defined by

$$r(t)\cdot t' := r(tt'). \tag{6}$$

We introduce in C the usual $\mathbb{C}$-linear basis $\langle r^{k,m,n} | k\in\mathbb{Z};\, m,n\in\mathbb{N}\rangle$, where

$$r^{k,m,n} = \begin{cases} r(t_{11}^k t_{12}^m t_{21}^n), & k\geq 0,\\ r(t_{12}^m t_{21}^n t_{22}^{-k}), & k<0. \end{cases} \tag{7}$$

In the following lemma we collect some useful relations concerning the $\mathcal{A}(\mathrm{U}_q(4))$-module structure of $\mathcal{A}(\mathrm{SU}_q(2))$.

Lemma 1. *The following relations are valid for $k \in \mathbb{Z}$ and $m, n \in \mathbb{N}$:*

$$
\begin{aligned}
r^{k,m,n} \cdot \begin{pmatrix} t_{11}^* & t_{12}^* \\ t_{21}^* & t_{22}^* \end{pmatrix} &= r^{k,m,n} \cdot \begin{pmatrix} t_{22} & -qt_{21} \\ -q^{-1}t_{12} & t_{11} \end{pmatrix}, \\
r^{k,m,n} \cdot \begin{pmatrix} t_{33}^* & t_{34}^* \\ t_{43}^* & t_{44}^* \end{pmatrix} &= r^{k,m,n} \cdot \begin{pmatrix} q^{m+n}t_{11} & q^{1-k}t_{12} \\ q^{-(1+k)}t_{21} & q^{-(m+n)}t_{22} \end{pmatrix} \\
&\quad - \theta(k)q(1-q^{-2k})r^{(k-1),m,n} \cdot \begin{pmatrix} 0 & 0 \\ 0 & t_{12}t_{21} \end{pmatrix} \\
&\quad - \theta(-k)q^{m+n-1}(1-q^{-2k})r^{(k+1),m,n} \cdot \begin{pmatrix} t_{12}t_{21} & 0 \\ 0 & 0 \end{pmatrix}, \\
r^{k,m,n} \cdot \begin{pmatrix} t_{33} & t_{34} \\ t_{43} & t_{44} \end{pmatrix} &= r^{k,m,n} \cdot \begin{pmatrix} q^{-(m+n)}t_{22} & -q^{-k}t_{21} \\ -q^{-k}t_{12} & q^{m+n}t_{11} \end{pmatrix} \\
&\quad - \theta(k)q(1-q^{-2k})r^{(k-1),m,n} \cdot \begin{pmatrix} t_{12}t_{21} & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad - \theta(-k)q^{m+n-1}(1-q^{-2k})r^{(k+1),m,n} \cdot \begin{pmatrix} 0 & 0 \\ 0 & t_{12}t_{21} \end{pmatrix},
\end{aligned}
$$

where $\theta(k) = 1$ for $k \geq 0$ and $\theta(k) = 0$ for $k < 0$.

Proof. They are obtained by using the following relations valid for $i < k,\ j < 1$:

$$
t_{ij}^n t_{kl} = t_{kl}t_{ij}^n - q^{-1}(1-q^{2n})t_{il}t_{kj}t_{ij}^{n-1}, \qquad t_{ij}t_{kl}^m = t_{kl}^m t_{ij} + q(1-q^{-2m})t_{il}t_{kj}t_{kl}^{m-1}. \tag{8}
$$

□

By construction

$$
\Delta_r := (\mathrm{id} \otimes r)\Delta : \mathcal{A}(\mathbb{S}_q^7) \to \mathcal{A}(\mathbb{S}_q^7) \otimes \mathcal{A}(\mathrm{SU}(2)_q)
$$

defines a right $\mathcal{A}(\mathrm{SU}(2)_q)$-coaction on $\mathcal{A}(\mathbb{S}_q^7)$. It satisfies $\Delta_r(uv) = \Delta_r(u)\Delta(v)$, for $u, v \in \mathcal{A}(\mathbb{S}_q^7)$. The space of functions on the quantum four-sphere Σ_q^4 is the space of coinvariants with respect to this coaction, i.e. $\mathcal{A}(\Sigma_q^4) = \{a \in \mathcal{A}(\mathbb{S}_q^7) | \Delta_r(a) = a \otimes r(1)\}$. The algebra $\mathcal{A}(\Sigma_q^4)$ is generated by $\{a, a^*, b, b^*, R\}$, where

$$
a = z_1 z_4^* - z_2 z_3^*, \qquad b = z_1 z_3 + q^{-1} z_2 z_4, \qquad R = z_1 z_1^* + z_2 z_2^*.
$$

They satisfy the following relations:

$$
\begin{aligned}
&Ra = q^{-2}aR, \qquad Rb = q^2 bR, \qquad ab = q^3 ba, \qquad ab^* = q^{-1}b^*a,\\
&aa^* + q^2 bb^* = R(1-q^2R), \qquad aa^* = q^2a^*a + (1-q^2)R^2,\\
&b^*b = q^4 bb^* + (1-q^2)R.
\end{aligned}
$$

3. The $\mathrm{SU}_q(2)$ principal bundle $\mathbb{S}^7_q$ over Σ^4_q

In this section we investigate if the structure introduced above forms a coalgebra–Galois extension, which is essential to define an (algebraic) quantum principal bundle.

Let C be a coalgebra, P a right C-comodule algebra with the multiplication $m_P : P\otimes P \to P$ and coaction $\Delta_R : P \to P \otimes C$. Let $B \subseteq P$ be the subalgebra of coinvariants, i.e. $B = \{b \in P|\Delta_R(bp) = b\Delta_R(p), \forall p \in P\}$. The canonical left P-linear right C-colinear map χ is defined by

$$\chi := (m_P \otimes \mathrm{id}) \circ (\mathrm{id} \otimes_B \Delta_R) : P \otimes_B P \to P \otimes C,\ p' \otimes_B p \mapsto p'\Delta_R p.$$

If χ is bijective one says that these data form a coalgebra–Galois extension (see [6]). In this case the translation map is defined as

$$\vartheta : C \to P \otimes_B P, \qquad \vartheta(h) = \sum_{[h]} h^{[1]} \otimes_B h^{[2]} := \chi^{-1}(1 \otimes h). \tag{9}$$

If C is a Hopf algebra and Δ_R is an algebra homomorphism, the above structure is called a Hopf–Galois extension. In this case the translation map ϑ is always determined by its values on the algebra generators of C; in fact one has that

$$\vartheta(hh') = \sum_{[h][h']} h'^{[1]}h^{[1]} \otimes_B h^{[2]}h'^{[2]}. \tag{10}$$

We specify now the above framework to our case of $P = \mathcal{A}(\mathbb{S}^7_q)$, $B = \mathcal{A}(\Sigma^4_q)$, $C = \mathcal{A}(\mathrm{SU}_q(2))$ and $\Delta_R = \Delta_r$. Since Δ_r is not an algebra homomorphism we are in the more general setting of coalgebra extensions. To answer the question of bijectivity of χ we cannot simply define the translation map on generators and then use formula (10) to extend it to the whole C. We shall instead generalize (10) by employing the $\mathcal{A}(\mathrm{U}_q(4))$-module structure defined in (6). By considering the coaction Δ of $\mathcal{A}(\mathrm{U}_q(4))$ on $\mathcal{A}(\mathbb{S}^7_q)$, we define the following right $\mathcal{A}(\mathbb{S}^7_q)$-module structure on $\mathcal{A}(\mathbb{S}^7_q) \otimes \mathcal{A}(\mathrm{SU}_q(2))$:

$$(u \otimes x) \triangleleft v = uv_{(0)} \otimes x \cdot v_{(1)}, \quad u, v \in \mathcal{A}(\mathbb{S}^7_q),\ \ x \in \mathcal{A}(\mathrm{SU}_q(2)),$$

where $\Delta(v) = \sum_{(v)} v_{(0)} \otimes v_{(1)}$.

By direct computation one can verify that χ is in fact also right $\mathcal{A}(\mathbb{S}^7_q)$-linear with respect to $\triangleleft$.

The following proposition is the main result of the paper.

Proposition 2. *Let* $C = \mathcal{A}(\mathrm{SU}_q(2)) \simeq \mathcal{A}(\mathrm{U}_q(4))/\mathcal{R}$, $P = \mathcal{A}(\mathbb{S}^7_q)$, $B = \mathcal{A}(\Sigma^4_q)$ *and* $\Delta_R = \Delta_r$ *be defined as above. Then the canonical map* χ *is bijective.*

Proof. First we prove the surjectivity of χ by giving a right inverse, i.e. a map $\tau : P\otimes C \to P\otimes_B P$ such that $\chi\circ\tau = \mathrm{id}$. We will define it on $1\otimes r^{k,m,n}$ and then extend it by left P-linearity on $P\otimes C$.

We give an iterative definition. Let $\tau(1\otimes r(1)) = 1\otimes_B 1$. Next assume that for any k, m, n such that $|k|+m+n \le N$, $\tau_{k,m,n} = \tau(1\otimes r^{k,m,n})$ is defined so that $\chi(\tau_{k,m,n}) = 1\otimes r^{k,m,n}$. We claim that (in partial analogy to (10)) we can define τ on all elements of the basis of degree $N+1$ by the following formulas:

$$\begin{aligned}
\tau_{k+1,m,n} &= q^{2+m+n}z_1^*\tau_{k,m,n}z_1 + q^{2+m+n}z_2\tau_{k,m,n}z_2^* + q^2 z_3\tau_{k,m,n}z_3^* \\
&\quad + z_4^*\tau_{k,m,n}z_4 \quad \text{for } k\ge 0,\\
\tau_{k-1,m,n} &= q^4 z_1\tau_{k,m,n}z_1^* + q^2 z_2^*\tau_{k,m,n}z_2 + q^{m+n}z_3^*\tau_{k,m,n}z_3 \\
&\quad + q^{m+n}z_4\tau_{k,m,n}z_4^* \quad \text{for } k\le 0,\\
\tau_{k,m+1,n} &= q^{2+((|k|-k)/2)}z_1^*\tau_{k,m,n}z_2 - q^{3+((|k|-k)/2)}z_2\tau_{k,m,n}z_1^* \\
&\quad + q^{1+((|k|+k)/2)}z_3\tau_{k,m,n}z_4^* - q^{(|k|+k)/2}z_4^*\tau_{k,m,n}z_3 \quad \text{for } k\in\mathbb{Z},\\
\tau_{k,m,n+1} &= -q^{3+((|k|-k)/2)}z_1\tau_{k,m,n}z_2^* + q^{2+((|k|-k)/2)}z_2^*\tau_{k,m,n}z_1 \\
&\quad - q^{(k+|k|)/2}z_3^*\tau_{k,m,n}z_4 + q^{1+((k+|k|)/2)}z_4\tau_{k,m,n}z_3^* \quad \text{for } k\in\mathbb{Z}.
\end{aligned} \tag{11}$$

The proof of the good definition of τ is postponed to Section 4. Let us apply χ to r.h.s. of the first equality of (11). The use of the left and right P-linearity of χ yields

$$\begin{aligned}
&\sum_{j=1}^{4}(q^{2+m+n}z_1^*z_j\otimes r^{k,m,n}\cdot t_{j1} + q^{2+m+n}z_2z_j^*\otimes r^{k,m,n}\cdot t_{j2}^* + q^2 z_3z_j^*\otimes r^{k,m,n}\cdot t_{j3}^* \\
&\qquad + z_4^*z_j\otimes r^{k,m,n}\cdot t_{j4}) = q^{m+n}(q^2z_1^*z_1 + q^2z_2z_2^* + q^2z_3z_3^* + z_4^*z_4)\otimes r^{k,m,n}\cdot t_{11} \\
&\qquad + (q^{2+m+n}z_1^*z_2 - q^{3+m+n}z_2z_1^* + q^{1-k}z_3z_4^* - q^{-k}z_4^*z_3)\otimes r^{k,m,n}\cdot t_{21} \\
&\quad = q^{m+n}\otimes r^{k,m,n}\cdot t_{11} = 1\otimes r^{k+1,m,n},
\end{aligned}$$

where in the penultimate equality we have used (4).

Similarly, χ applied to r.h.s. of the second equality of (11) yields ($k<0$)

$$\begin{aligned}
&(q^4z_1^*z_1 + q^2z_2^*z_2 + z_3^*z_3 + z_4z_4^*)\otimes r^{k,m,n}\cdot t_{22} + (-q^3z_1z_2^* + q^2z_2^*z_1 \\
&\quad - q^{m+n-k}z_3^*z_4 + q^{1+m+n-k}z_4z_3^*)\otimes r^{k,m,n}\cdot t_{12} = 1\otimes r^{k,m,n}\cdot t_{22} = 1\otimes r^{k-1,m,n}.
\end{aligned}$$

Next, the r.h.s. of the third equality of (11), after application of χ yields

$$\begin{aligned}
&q^{(|k|-k)/2}(q^2z_1^*z_1 + q^2z_2z_2^* + q^2z_3z_3^* + z_4^*z_4)\otimes r^{k,m,n}\cdot t_{12} \\
&\quad + [q^{2+((|k|-k)/2)}(z_1^*z_2 - qz_2z_1^*) + q^{((k+|k|)/2)-m-n}(qz_3z_4^* - z_4^*z_3)]\otimes r^{k,m,n}\cdot t_{22} \\
&\quad + \theta(k)q^{1+((k+|k|)/2)}(q^{-2k}-1)(qz_3z_4^* - z_4^*z_3)\otimes r^{k-1,m,n}\cdot t_{12}t_{21} = 1\otimes r^{k,m+1,n}.
\end{aligned}$$

Finally, χ applied to r.h.s. of the fourth equality of (11) yields

$$\begin{aligned} & q^{(|k|-k)/2}(q^4 z_1^* z_1 + q^2 z_2^* z_2 + z_3^* z_3 + z_4 z_4^*) \otimes r^{k,m,n} \cdot t_{21} \\ & \quad + q^{(|k|-k)/2}(-q^3 z_1 z_2^* + q^2 z_2^* z_1 - q^{k+m+n} z_3^* z_4 + q^{1+k+m+n} z_4 z_3^*) \otimes r^{k,m,n} \cdot t_{11} \\ & \quad + \theta(-k) q^{(|k|+k)/2} q^{m+n}(1-q^{-2k})(q^{-1} z_3^* z_4 - z_4 z_3^*) \otimes r^{k+1,m,n} \cdot t_{12} t_{21} \\ & = 1 \otimes r^{k,m,n+1}. \end{aligned}$$

The map τ is then defined by giving its action on the basis and it satisfies $\chi \circ \tau = \mathrm{id}$ so that χ is surjective.

Injectivity of χ is a consequence of the P-linearity of the map τ, whose proof is postponed to the next section. In fact we have that $\tau(\chi(a \otimes b)) = \tau(a\chi(1 \otimes 1) \triangleleft b) = a\tau(1 \otimes r(1))b = a \otimes b$, so that $\tau = \chi^{-1}$. □

4. Proof of good definition and P-linearity of τ

The relations (11) provide an iterative definition of the map τ once we prove that $\tau_{k,m,n}$ is uniquely defined starting from $\tau_{k-1,m,n}, \tau_{k,m-1,n}, \tau_{k,m,n-1}$. We will prove the good definition of τ on the linear basis $r^{k,m,n}$ by induction on $|k| + m + n$. The P-linearity will come as an easy consequence of the induction procedure. Since computations are quite heavy and long we will limit ourselves to sketch the proof.

We suppose that $\tau(1 \otimes r^{k,m,n}) = \tau_{k,m,n}$ is well defined for $|k| + m + n \leq N$ and that the following relations are true for $|k| + m + n \leq N - 1$:

$$\left.\begin{aligned} & q^{-(m+n)} z_1 \tau_{k+1,m,n} + z_2 \tau_{k,m,n+1} = \tau_{k,m,n} z_1, \\ & -q^{-k} z_3 \tau_{k,m,n+1} + z_4 \tau_{k+1,m,n} = \tau_{k,m,n} z_4, \\ & -q z_1^* \tau_{k,m,n+1} + z_2^* q^{-(m+n)} \tau_{k+1,m,n} = \tau_{k,m,n} z_2^*, \\ & z_3^* \tau_{k+1,m,n} + z_4^* q^{-(1+k)} \tau_{k,m,n+1} = \tau_{k,m,n} z_3^*, \end{aligned}\right\} \quad k \geq 0, \tag{12}$$

$$\left.\begin{aligned} & q^{-(m+n)} z_3 \tau_{k-1,m,n} - z_4 \tau_{k,m+1,n} = \tau_{k,m,n} z_3, \\ & q^k z_1 \tau_{k,m+1,n} + z_2 \tau_{k-1,m,n} = \tau_{k,m,n} z_2, \\ & q z_3^* \tau_{k,m+1,n} + z_4^* q^{-(m+n)} \tau_{k-1,m,n} = \tau_{k,m,n} z_4^*, \\ & z_1^* \tau_{k-1,m,n} - z_2^* q^{-(1-k)} \tau_{k,m+1,n} = \tau_{k,m,n} z_1^*, \end{aligned}\right\} \quad k \leq 0, \tag{13}$$

and the following ones for $|k| + m + n \leq N$:

$$\left.\begin{aligned} & q^{m+n} z_2 \tau_{k-1,m,n} + q^{m+n+1} \tau_{k-1,m+1,n} z_1 = \tau_{k,m,n} z_2, \\ & z_3 \tau_{k-1,m,n} - q^{-k} \tau_{k-1,m+1,n} z_4 = \tau_{k,m,n} z_3, \\ & q^{m+n} z_1^* \tau_{k-1,m,n} - q^{m+n} \tau_{k-1,m+1,n} z_2^* = \tau_{k,m,n} z_1^*, \\ & z_4^* \tau_{k-1,m,n} + q^{1-k} \tau_{k-1,m+1,n} z_3^* = \tau_{k,m,n} z_4^*, \end{aligned}\right\} \quad k \geq 1 \tag{14}$$

$$\begin{aligned}
&q^{m+n}z_4\tau_{k+1,m,n} - q^{m+n+1}\tau_{k+1,m,n+1}z_3 = \tau_{k,m,n}z_4,\\
&z_1\tau_{k+1,m,n} + q^k\tau_{k+1,m,n+1}z_2 = \tau_{k,m,n}z_1, \qquad\qquad k \le -1\\
&q^{m+n}z_3^*\tau_{k+1,m,n} + q^{m+n}\tau_{k+1,m,n+1}z_4^* = \tau_{k,m,n}z_3^*,\\
&z_2^*\tau_{k+1,m,n} - q^{1+k}\tau_{k+1,m,n+1}z_1^* = \tau_{k,m,n}z_2^*.
\end{aligned} \tag{15}$$

It is a straightforward computation to verify the induction hypothesis for $N = 1$ and that (12), (13), (14), (15) imply the good definition of $\tau_{k,m,n}$ for $|k| + m + n = N + 1$.

In the following subsections we sketch the proof of (12) and (14). The set of equations (13) and (15) are proven along the same lines.

The left P–linearity of the map τ is clear by construction. The right P–linearity is a direct consequence of equations (12), (13), (14), (15) as can be seen by writing explicitly $\tau((1 \otimes r^{k,m,n}) \triangleleft z_\ell) = \tau(1 \otimes r^{k,m,n})z_\ell$ for each ℓ and the corresponding one for z_ℓ^*.

For each $n \in \mathbb{Z}$ let us define $a_n = z_1z_4^* - q^nz_2z_3^*$ and $b_n = z_1z_3 + q^{n-1}z_2z_4$.

4.1. *Proof of (12)*

The following lemma is preliminary to get the result.

Lemma 2. *Relations (12) are true if and only if the following relations hold for $k \ge 0$:*

$$\begin{aligned}
&(1 - q^2R)\tau_{k,m,n}z_1 = q^{2-m-n}b_{m+n+k}\tau_{k,m,n}z_3^* + q^{-m-n}a_{k+m+n}\tau_{k,m,n}z_4,\\
&R\tau_{k,m,n}z_4 = q^{2-k}b_{k+m+n}\tau_{k,m,n}z_2^* + q^{2+m+n}a^*_{-(m+n+k)}\tau_{k,m,n}z_1,\\
&(1 - q^2R)\tau_{k,m,n}z_2^* = q^kb^*_{-(m+n+k)}\tau_{k,m,n}z_4 - q^{3+k}a^*_{-(m+n+k)}\tau_{k,m,n}z_3^*,\\
&q^4R\tau_{k,m,n}z_3^* = q^{2+m+n}b^*_{-(m+n+k)}\tau_{k,m,n}z_1 - q^{3-k}a_{k+m+n}\tau_{k,m,n}z_2^*.
\end{aligned} \tag{16}$$

Proof. In order to go from (12) to (16), substitute in each of (12) the iterative definition (11). All the steps can be retraced back to go in the opposite direction. □

Lemma 3. *As a consequence of the induction hypothesis, for each (k, m, n) such that $k + m + n = N - 1$ we have*

$$\begin{aligned}
&(1 - q^2R)\tau_{k+1,m,n} = z_4^*\tau_{k,m,n}z_4 + q^2z_3\tau_{k,m,n}z_3^*,\\
&b_{k+m+n+1}\tau_{k+1,m,n}q^{-m-n} = qz_3\tau_{k,m,n}z_1 + q^kz_2\tau_{k,m,n}z_4,\\
&a_{k+m+n+1}\tau_{k+1,m,n}q^{-m-n} = q^{-1}z_4^*\tau_{k,m,n}z_1 - q^{1+k}z_2\tau_{k,m,n}z_3^*,\\
&a^*_{-(m+n+k+1)}\tau_{k+1,m,n} = q^{-1}z_1^*\tau_{k,m,n}z_4 - q^{-k-1}z_3\tau_{k,m,n}z_2^*,\\
&R\tau_{k+1,m,n} = q^{m+n}(z_1^*\tau_{k,m,n}z_1 + z_2\tau_{k,m,n}z_2^*),\\
&b^*_{-(m+n+k+1)}\tau_{k+1,m,n} = qz_1^*\tau_{k,m,n}z_3^* + q^{-k-2}z_4^*\tau_{k,m,n}z_2^*.
\end{aligned} \tag{17}$$

Proof. For instance, in order to get the first one, multiply on the left by q^2z_3 and by z_4^* the fourth and the second equation of (12), respectively; then add and collect the terms. All the other relations are obtained in a similar way. □

Let (k, m, n) such that $k+m+n=N$ and apply (17) for $(k-1, m, n)$. For example, in order to prove the first of (16):

$$\begin{aligned}(1-q^2R)\tau_{k,m,n}z_1 &= (z_4^*\tau_{k-1,m,n}z_4 + q^2 z_3\tau_{k-1,m,n}z_3^*)z_1\\ &= (q^{-1}z_4^*\tau_{k-1,m,n}z_1 - q^k z_2\tau_{k-1,m,n}z_3^*)z_4\\ &\quad + q^2(q^{k-1}z_2\tau_{k-1,m,n}z_4 + qz_3\tau_{k-1,m,n}z_1)z_3^*\\ &= q^{-m-n}(a_{k+m+n}\tau_{k,m,n}z_4 + q^2 b_{k+m+n}\tau_{k,m,n})z_3^*,\end{aligned}$$

where the first and third lines are obtained by using (17) for $(k-1, m, n)$. All the other equations (16) are obtained in a similar way. Using Lemma 1 the (12) are then proved.

4.2. *Proof of (13)*

In order to prove them we use the strategy adopted for (12). By substituting in (14) the iterative definition of τ we get the following Lemma.

Lemma 4. *Relations (14) are true if and only if the following equations hold for* $k \geq 1$

$$\begin{aligned}&z_2\tau_{k-1,m,n}(1-q^2R) = -q^{k+2}z_3\tau_{k-1,m,n}a_{-(k+m+n-1)} + q^{k-1}z_4^*\tau_{k-1,m,n}b_{-(k+m+n-1)},\\ &q^2 z_3\tau_{k-1,m,n}R = q^{m+n}z_1^*\tau_{k-1,m,n}b_{-(m+n+k-1)} - q^{2-k}z_2\tau_{k-1,m,n}a^*_{m+n+k-1},\\ &q^{m+n}z_1^*\tau_{k-1,m,n}(1-q^2R) = q^2 z_3\tau_{k-1,m,n}b^*_{k+m+n-1} + z_4^*\tau_{k-1,m,n}a^*_{m+n+k-1},\\ &z_4^*\tau_{k-1,m,n}R = q^{2+m+n}z_1^*\tau_{k-1,m,n}a_{-(m+n+k-1)} + q^{3-k}z_2\tau_{k-1,m,n}b^*_{m+n+k-1}.\end{aligned}\tag{18}$$

We have the following Lemma:

Lemma 5. *As a consequence of the induction hypothesis, for each* $k+m+n=N-1$ *we have*

$$\begin{aligned}&\tau_{k,m,n}(1-q^2R) = z_4^*\tau_{k-1,m,n}z_4 + q^2 z_3\tau_{k-1,m,n}z_3^*,\\ &\tau_{k,m,n}R = q^{m+n}(z_2\tau_{k-1,m,n}z_2^* + z_1^*\tau_{k-1,m,n}z_1),\\ &\tau_{k,m,n}a_{-(k+m+n)} = q^{-1}z_4^*\tau_{k-1,m,n}z_1 - q^{-k}z_2\tau_{k-1,m,n}z_3^*,\\ &\tau_{k,m,n}b_{-(k+m+n)} = qz_3\tau_{k-1,m,n}z_1 + q^{-k-1}z_2\tau_{k-1,m,n}z_4,\\ &\tau_{k,m,n}a^*_{k+m+n} = q^{m+n-1}z_1^*\tau_{k-1,m,n}z_4 - q^{k+m+n}z_3\tau_{k-1,m,n}z_2^*,\\ &\tau_{k,m,n}b^*_{k+m+n} = q^{m+n+1}z_1^*\tau_{k-1,m,n}z_3^* + q^{k+m+n-1}z_4^*\tau_{k-1,m,n}z_2^*.\end{aligned}\tag{19}$$

Proof. For example, in order to get the first relation, multiply on the right by z_2^* and by z_1 the first and the third of (14), respectively, and then add. The other relations are shown in a similar way. □

Let (k, m, n) be such that $k + m + n = N$ and let us apply (19) for $(k - 1, m, n)$. For example, in order to show the first of (19):

$$\begin{aligned} z_2\tau_{k-1,m,n}(1 - q^2R) &= z_2z_4^*\tau_{k-2,m,n}z_4 + q^2z_2z_3\tau_{k-2,m,n}z_3^* \\ &= q^{k-1}z_4^*(q^{-k}z_2\tau_{k-2,m,n}z_4 + qz_3\tau_{k-2,m,n}z_1) \\ &\quad - q^{k+2}z_3(-q^{1-k}z_2\tau_{k-2,m,n}z_3^* + q^{-1}z_4^*\tau_{k-2,m,n}z_1) \\ &= q^{k-1}z_4^*\tau_{k-1,m,n}b_{-(k+m+n-1)} - q^{k+2}z_3\tau_{k-1,m,n}a_{-(k-1+m+n)}, \end{aligned}$$

where we used the first, the third and the fourth of the (19) for $(k - 1, m, n)$. All the others relations are obtained in a similar way. This concludes the proof of (14).

5. Conclusions

The bijectivity of the canonical map is a key result to study the non-commutative geometry of this fibration.

To this purpose it is relevant the notion of *principal Galois extension* introduced in [7,14] which consists of a coalgebra Galois extension with the following additional requirements (the notations are the same as in the beginning of Section 3): (i) the entwining map $\psi : C\otimes P \to P\otimes C, c\otimes p \to \chi(\chi^{-1}(1\otimes c)p)$ is bijective; (ii) there exists a group like element $e \in C$ such that $\Delta_R(p) = \psi(e \otimes p)$; (iii) there exists a strong connection.

In our case the points (i) and (ii) are easily satisfied. In fact $\psi(c \otimes P) = (1 \otimes c) \triangleleft p$ can be inverted by $\psi^{-1}(p\otimes c) = c \cdot S^{-1}(p_{(1)}) \otimes p_{(0)}$ and $e = r(1)$. It is crucial that $P = \mathcal{A}(\mathbb{S}_q^7)$ is a $\mathcal{A}(\mathrm{U}_q(4))$-comodule and $C = \mathcal{A}(\mathrm{SU}_q(2))$ a $\mathcal{A}(\mathrm{U}_q(4))$-module.

A result (Theorem 2.44) contained in the paper in preparation [16] solves the point (iii): if C is a cosemisimple coalgebra, right H-module quotient of an Hopf algebra H with bijective antipode then the surjectivity of the canonical map implies its injectivity and the existence of a strong connection (or equivariant projectivity in their terminology).

The translation map ϑ can be lifted to $\ell : \mathcal{A}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathbb{S}_q^7) \otimes \mathcal{A}(\mathbb{S}_q^7)$ by making use of the relations (11). We conjecture that the lifted map ℓ is bicolinear with respect to $(1\otimes \Delta_r)$ and $(\bar{\Delta}_r \otimes 1)$ where $\bar{\Delta}_r : p \to r(S^{-1}(p_{(1)})) \otimes p_{(0)}$ [14], so that it gives an explicit expression to the strong connection.

A relevant consequence of principality is that it makes possible, in the quantum setting, the construction of associated vector bundles. Given, in fact, any finite dimensional left corepresentation of C, $(\rho : V \to C \otimes V)$, the cotensor product $P\square_\rho V$ turns out to be a finitely generated and projective B-module.

In our case for any corepresentation of spin j of $\mathcal{A}(\mathrm{SU}_q(2))$ one has a *quantum vector bundle* and the corresponding class in the positive cone of $K_0(\mathcal{A}(\Sigma_q^4))$.

In [2] it was already computed the vector bundle associated to the spin $\frac{1}{2}$ corepresentation and its pairing with the K-homology generators. The non-triviality of this paring allows to conclude that the coalgebra Galois extension is non-cleft. Using the formula for the Chern–Connes character given in [7] we have all we need to compute the Chern numbers of the generic associated bundles. This will be the content of a future work.

Acknowledgements

We want to thank P.M. Hajac for enlightening discussions, for explaining us his results in [7] and for providing us useful bibliographical informations, and P. Schauenburg and H.-J. Schneider for anticipating us the results contained in [16].

References

[1] F. Bonechi, N. Ciccoli, R. Giachetti, E. Sorace, M. Tarlini, Unitarity of induced representations from coisotropic quantum subgroups, Lett. Math. Phys. 49 (1999) 17–31.
[2] F. Bonechi, N. Ciccoli, M. Tarlini, Non-commutative instantons on the 4-sphere from quantum groups, Commun. Math. Phys. 226 (2002) 419–432.
[3] F. Bonechi, N. Ciccoli, M. Tarlini, Quantum 4-sphere: The Infinitesimal Approach, Banach Center Publications, in press.
[4] T. Brzeziński, Quantum homogeneous spaces as quantum quotient spaces, J. Math. Phys. 37 (1996) 2388–2399.
[5] T. Brzeziński, On modules associated to coalgebra Galois extensions, J. Algebra 215 (1999) 290–317.
[6] T. Brzeziński, P.M. Hajac, Coalgebra extensions and algebra coextensions of Galois type, Commun. Algebra 27 (1999) 1347–1367.
[7] T. Brzeziński, P.M. Hajac, The Chern–Galois character. math.KT/0306436.
[8] T. Brzeziński, S. Majid, Quantum group gauge theory on quantum spaces, Commun. Math. Phys. 157 (1993) 591–638 (Erratum 167 (1995) 235).
[9] T. Brzeziński, S. Majid, Coalgebra bundles, Commun. Math. Phys. 191 (1998) 467–492.
[10] N. Ciccoli, Quantization of coisotropic subgroups, Lett. Math. Phys. 42 (1997) 23–38.
[11] L. Dabrowski, The Garden of Quantum Spheres, Banach Center Publications, in press. math.QA/0212264.
[12] M. Durdević, Geometry of quantum principal bundles I, Commun. Math. Phys. 175 (1996) 427–521.
[13] P.M. Hajac, Bundles over quantum spheres and non-commutative index theorem, K-Theory 21 (2000) 141–150.
[14] P.M. Hajac, R. Matthes, W. Szymański, Chern numbers for two families of noncommutative Hopf fibrations, C.R.A.S. Série I, in press. math.QA/0302256.
[15] A. Klimyk, K. Schmüdgen, Quantum Groups and Their Representations, Springer, Berlin, 1997.
[16] P. Schauenburg, H.-J. Schneider, Galois type extensions of noncommutative algebras, in preparation.
[17] H.-J. Schneider, Principal homogeneous spaces for arbitrary Hopf algebras, Isr. J. Math. 72 (1990) 167–195.
[18] L.L. Vaksman, Ya. Soibelman, Algebra of functions on the quantum group $\mathrm{SU}(N+1)$ and odd dimensional quantum spheres, Leningrad Math. J. 2 (1991) 1023–1042.